SHORE

Examination Number:

Set:

Year 12
HSC Assessment Task 5 - Trial HSC
17th August 2017

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A NESA reference sheet is provided
- Answer Questions 1–10 on the Multiple Choice Answer Sheet provided
- Start each of Questions 11–14 in a new writing booklet
- In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.
- Write your examination number on the front cover of each booklet
- If you do not attempt a question, submit a blank booklet marked your examination number and “N/A” on the front cover

Total marks – 70

Section I Pages 3–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–12

60 marks

- Attempt Questions 11–14
- Allow about 1 hour 45 minutes for this section

Note: Any time you have remaining should be spent revising your answers.

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1–10.

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1 A function is represented by the parametric equations
$x = 2t + 1$ and $y = t - 2$.
Which of the following is the Cartesian equation for the function?

(A) $x - 2y + 3 = 0$

(B) $x - 2y - 3 = 0$

(C) $x + 2y + 5 = 0$

(D) $x - 2y - 5 = 0$

2 Given that $\sin\alpha = \dfrac{4}{5}$ and $\cos\beta = \dfrac{5}{13}$ and both α and β are acute, what is the exact value of $\cos(\alpha - \beta)$?

(A) $\dfrac{-33}{65}$

(B) $\dfrac{27}{65}$

(C) $\dfrac{56}{65}$

(D) $\dfrac{63}{65}$

3 What is the size of the acute angle, to the nearest degree, between the lines
$2x - 3y + 4 = 0$ and $x + 2y - 7 = 0$?

(A) 7°

(B) 19°

(C) 41°

(D) 60°

4 What is the Domain and Range of $y = 3\cos^{-1} 2x$?

(A) Domain: $-\frac{1}{2} \le x \le \frac{1}{2}$ and Range: $0 \le y \le 3\pi$.

(B) Domain: $-1 \le x \le 1$ and Range: $-\pi \le y \le \pi$.

(C) Domain: $-\frac{1}{2} \le x \le \frac{1}{2}$ and Range: $-\pi \le y \le \pi$.

(D) Domain: $-1 \le x \le 1$ and Range: $0 \le y \le \frac{\pi}{3}$.

5 What is the value of $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4 - x^2}}$?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$

(D) $\frac{2\pi}{3}$

6 The expression $\sin 4x + \sqrt{3}\cos 4x$ can be written in the form $2\sin(4x + \alpha)$.
What is the value of α ?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

7 A particle is moving in simple harmonic motion with displacement x.
The acceleration a of the particle is given by

$$a = 25 - 5x.$$

What is the period of motion?

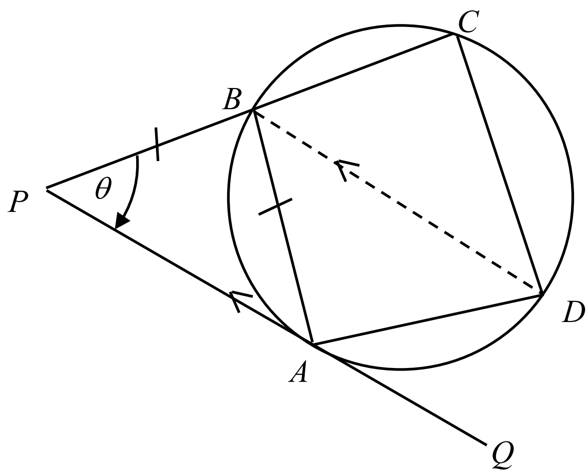
(A) $\sqrt{5}$

(B) 5

(C) $\dfrac{2\pi}{\sqrt{5}}$

(D) $\dfrac{2\pi}{5}$

8 In the diagram $ABCD$ is a cyclic quadrilateral. The tangent PQ touches the circle at A. The diagonal BD is parallel to the tangent PQ. QA produced intersects with CB produced at P. $BP = BA$ and $\angle BPA = \theta$.



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What is the size of $\angle BCD$?

(A) θ

(B) 2θ

(C) $180° - \theta$

(D) $180° - 2\theta$

9 Which of the following is the general solution of $2\cos 2x = 1$?

(A) $2n\pi \pm \dfrac{\pi}{6}$

(B) $2n\pi \pm \dfrac{\pi}{3}$

(C) $n\pi \pm \dfrac{\pi}{6}$

(D) $n\pi \pm \dfrac{\pi}{3}$

10 Given that the roots of the cubic equation $4x^3 - 3x^2 - 5x + 2 = 0$ are α, β and γ, what is the value of $\alpha^2 + \beta^2 + \gamma^2$?

(A) $\dfrac{1}{4}$

(B) $\dfrac{1}{2}$

(C) $\dfrac{9}{16}$

(D) $\dfrac{49}{16}$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

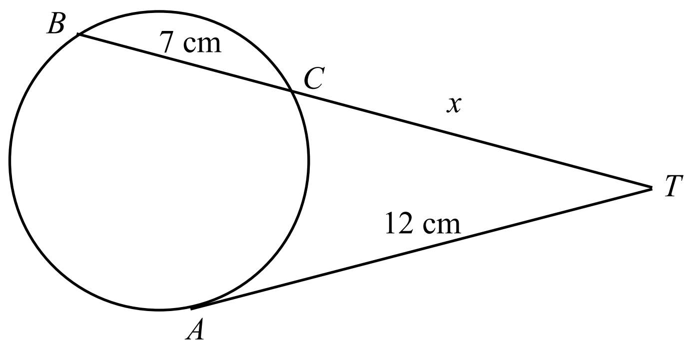
In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Solve the inequality $\dfrac{x}{x-4} \geq 2$. **2**

(b) The point P divides the interval AB externally in the ratio 3 : 1. **2**
Given the points $A(-3,1)$ and $B\ (1,-2)$, find the coordinates of P.

(c) In the diagram below, AT is the tangent to the circle at A. **2**
BT is a secant meeting the circle at B and C.



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Given that $AT = 12$ cm, $BC = 7$ cm and $CT = x$ cm, find the value of x.
No reasons are required.

(d) Use the substitution $x = u - 2$ to evaluate $\displaystyle\int_{-1}^{2} \frac{3x+5}{\sqrt{x+2}}\, dx$. **3**

Question 11 continues on the following page

(e) The function $f(x) = \log_e x - \sin x + 1$ has a zero near $x = 0.75$. **3**
Use one application of Newton's Method to obtain another approximation to this zero.

Give your answer correct to 2 significant figures.

(f) Find the term independent of x in the expansion of $\left(x^2 - \dfrac{2}{x}\right)^{12}$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) The polynomial $P(x) = 2x^3 + x^2 + ax + 6$ has a zero at $x = 2$.

(i) Determine the value of a. **1**

(ii) Find the linear factors of $P(x)$. **2**

(iii) Hence, or otherwise, solve $P(x) \geq 0$ **1**

(b) Prove by Mathematical Induction that

$2 \times 1! + 5 \times 2! + 10 \times 3! + ... + (n^2 + 1)n! = n \times (n+1)!$ for $n \geq 1$. **3**

(c) Consider the function $f(x) = 6x - 2x^3$.

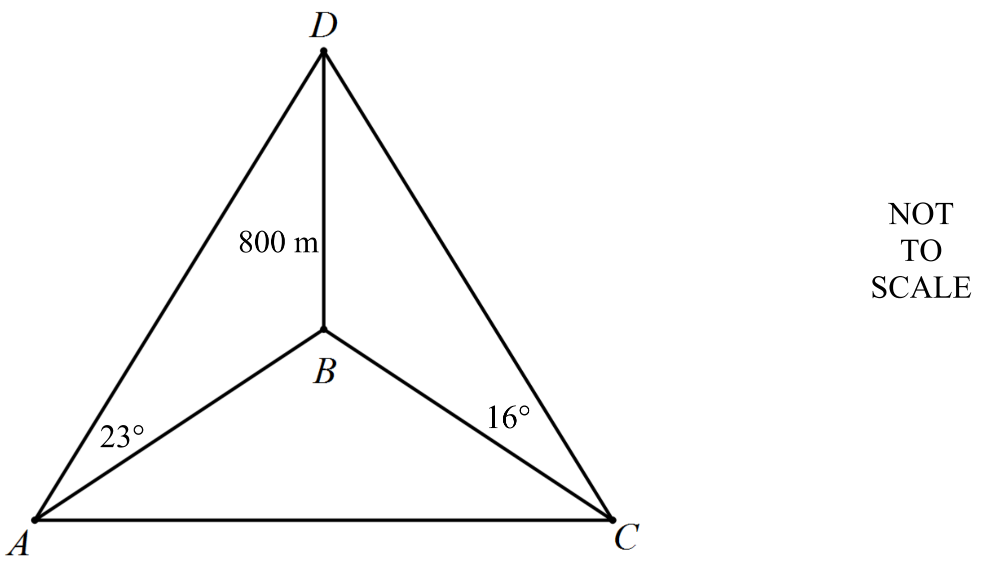
(i) Find the largest domain containing the origin for which $f(x)$ has an inverse function $f^{-1}(x)$. **2**

(ii) Find the gradient of the inverse function at $x = 0$. **2**

Question 12 continues on the following page

Question 12 (continued)

(d) Chris and Aaron are competing in a sailing boat race. Chris (C) can see the top of a vertical cliff (D) that is 800 m above sea level. The cliff is on a bearing of $329°$ from his position and the angle of elevation to the top of the cliff (D) is $16°$.
Aaron (A) can also see the top of the cliff on a bearing of $049°$ with an angle of elevation of $23°$.
The base of the cliff (B) is at sea level.



(i) Show that $\angle ABC = 80^\circ$. **1**

(ii) Find the distance AC between the two sailing boats to the nearest metre. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Consider the point $P(2p, p^2)$ that lies on the parabola $x^2 = 4y$.

(i) Show that the equation of the normal at P is given by **2**

$$x + py - 2p - p^3 = 0.$$

(ii) The normal meets the y-axis at Q. **2**
Find the coordinates of the midpoint M of PQ.

(iii) Find the locus of the point M. **1**

(b) The acceleration of a particle moving in a straight line is given by

$$a = 2x^3 + 2x$$

where x is the displacement of the particle from the origin at time t seconds.

Initially the particle is at the origin moving at 1 m/s.

(i) Show that the velocity of the particle is given by $v = x^2 + 1$. **2**

(ii) Hence, or otherwise, find the displacement of the particle after $\frac{\pi}{4}$ seconds. **3**

(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$. **2**

(d) A rectangle is expanding in such a way that at all times, its length is twice as long as its width. If its area is increasing at a rate of 18 cm^2/s, find the rate at which its perimeter is increasing when the width of the rectangle is 80 centimetres. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) A frozen cake is removed from a freezer at $-10°C$ and is placed in a room at a constant temperature of $20°C$.
Thereafter its temperature $T°$ is changing so that after t minutes

$$\frac{dT}{dt} = K(20 - T) \text{ where } K \text{ is a constant.}$$

(i) Show that $T = 20 - Be^{-Kt}$ satisfies this condition. **1**

(ii) Find the value of B. **1**

(iii) If, initially, the temperature was increasing at the rate of $3°C$ per minute, find the value of K. **2**

(iv) Find the temperature of the cake 5 minutes after it was placed in the room.
Give your answer to the nearest degree. **2**

(b) A football is kicked at an angle of α to the horizontal. The position of the ball at time t seconds is given by

$$x = vt\cos\alpha \quad \text{and}$$
$$y = vt\sin\alpha - \frac{1}{2}gt^2$$

(DO **NOT** PROVE THESE)

where g m/s^2 is the acceleration due to gravity and v m/s is the initial velocity of the football.

(i) Show that the equation of the path of the football is **2**

$$y = x\tan\alpha - \frac{gx^2}{2v^2}\sec^2\alpha .$$

(ii) If $g = 10$ m/s^2, v=20 m/s and the ball just clears the head of a 1.8 metre tall player that is 10 metre away, calculate the angle(s) to the horizontal at which the football is initially kicked.
Give your answer correct to the nearest minute. **3**

(c) By considering the expansion of $(1 + x)^n$ and the value of $\int_0^3 (1 + x)^n \, dx$, **4**

show that $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} 3^{k+1} = \frac{1}{n+1}(4^{n+1} - 1)$.

END OF PAPER

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1. $x = 2t + 1$

$y = t - 2$

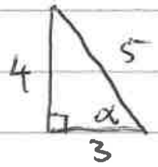
$\therefore t = y + 2$

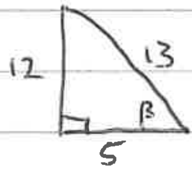
$x = 2(y+2) + 1$

$x = 2y + 4 + 1$

$x - 2y - 5 = 0$ (D)

2.





$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13}$

$= \frac{63}{65}$ (D)

3. $m_1 = \frac{2}{3}$ $m_2 = -\frac{1}{2}$

$\therefore \tan\theta = \left| \dfrac{\frac{2}{3} + \frac{1}{2}}{1 + \left(\frac{2}{3}\right)\left(-\frac{1}{2}\right)} \right|$

$\therefore \theta = 60^\circ$ (D)

4. $\frac{y}{3} = \cos^{-1} 2x$

$-1 \leq 2x \leq 1 \longrightarrow -\frac{1}{2} \leq x \leq \frac{1}{2}$

A

$0 \leq \frac{y}{3} \leq \pi \longrightarrow 0 \leq y \leq 3\pi$

5. $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\left(\frac{x}{2}\right) \right]_0^{\sqrt{3}}$

$= \frac{\pi}{3}$ B

6. $2\sin(4x + \alpha)$

$= 2\sin 4x \cos\alpha + 2\sin\alpha \cos 4x$

$\therefore 2\cos\alpha = 1$ $\qquad 2\sin\alpha = \sqrt{3}$

$\cos\alpha = \frac{1}{2}$ $\qquad \sin\alpha = \frac{\sqrt{3}}{2}$

$\therefore$ $\alpha = \frac{\pi}{3}$ C

7. $a = -5(x - 5)$

$\therefore -n^2 = -5$

$n = \sqrt{5}$

$\therefore P = \frac{2\pi}{\sqrt{5}}$ C

8. B

9. $2\cos 2x = 1$

$\cos 2x = \frac{1}{2}$

$\therefore 2x = 2\pi n \pm \frac{\pi}{3}$

$\therefore x = \pi n \pm \frac{\pi}{6}$ C

10. $4x^3 - 3x^2 - 5x + 2 = 0$

$(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= \left(\frac{3}{4}\right)^2 - 2\left(\frac{-5}{4}\right)$

$= \frac{49}{16}$ D

11. a) $\dfrac{x}{x-4} \geqslant 2$

critical points

$\boxed{x \neq 4}$

$$\frac{x}{x-4} = 2$$

$$x = 2x - 8$$

$$x = 8$$

test $x = 0$

$\dfrac{0}{-4} \geqslant 2$ FALSE

$\therefore \quad 4 < x \leq 8$

b) $3 : -1$

$$x = \frac{-1 \times -3 + 3 \times 1}{2}$$

$$= 3$$

$$y = \frac{-1 \times 1 + 3 \times -2}{2}$$

$$= -3\tfrac{1}{2}$$

$P\left(3, -3\tfrac{1}{2}\right)$

c) $12^2 = x(x+7)$

$144 = x^2 + 7x$

$0 = x^2 + 7x - 144$

$0 = (x+16)(x-9)$

$\therefore x = 9 \qquad (x > 0)$

d) $x = u - 2$

$\therefore \frac{dx}{du} = 1 \qquad dx = du$

$u = x + 2 \qquad x = 2 \rightarrow u = 4$

$x = -1 \rightarrow u = 1$

$3x + 5 = 3(u-2) + 5$

$= 3u - 6 + 5$

$= 3u - 1$

$\therefore \int_{+1}^{4} (3u - 1) u^{-\frac{1}{2}} \, du$

$= \int_{1}^{4} 3u^{\frac{1}{2}} - u^{-\frac{1}{2}} \, du$

$= \left[2u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_1^4$

$= (16 - 4) - (2 - 2)$

$= 12$

e) $f'(x) = \frac{1}{x} - \cos x$

$$\therefore x_2 = 0.75 - \frac{\log_e(0.75) - \sin(0.75) + 1}{\frac{1}{0.75} - \cos(0.75)}$$

$= 0.699\ldots$
$= 0.70$

f) $\text{Constant} = \binom{12}{4}(x^2)^4\left(-\frac{2}{x}\right)^8$

$= 126720$

12. (i) $0 = 2(2)^3 + 2^2 + 2a + 6$

$0 = 16 + 4 + 2a + 6$

$0 = 26 + 2a$

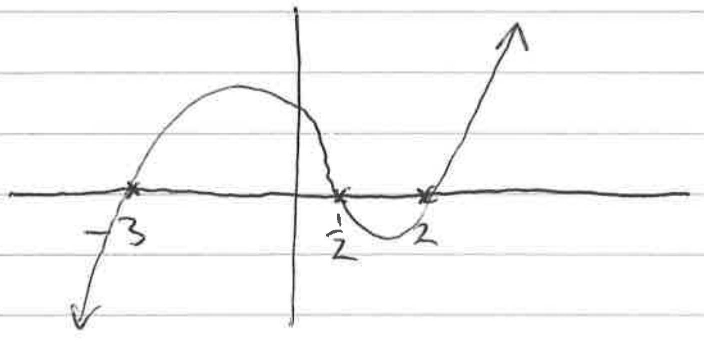
$\therefore a = -13$

(ii) $P(x) = 2x^3 + x^2 - 13x + 6$

$$\begin{array}{r l}
 & 2x^2 + 5x - 3 \\
x - 2 & \overline{)\, 2x^3 + x^2 - 13x + 6} \\
 & 2x^3 - 4x^2 \\
\hline
 & \quad 5x^2 - 13x + 6 \\
 & \quad 5x^2 - 10x \\
\hline
 & \qquad -3x + 6 \\
 & \qquad -3x + 6 \\
\hline
 & \qquad\quad 0
\end{array}$$

$\therefore P(x) = (x-2)(x+3)(2x-1)$

(iii)



$P(x) \geq 0 \qquad -3 \leq x \leq \frac{1}{2}, \quad x \geq 2.$

b) Prove true for $n=1$.

$LHS = 2 \times 1!$

$= 2$

$RHS = 1 \times (1+1)!$

$= 2$

$\therefore$ true for $n=1$.

Assume true for $n=k$.

$2 \times 1! + 5 \times 2! + \dots (k^2+1)k! = k \times (k+1)!$

If true for $n=k$, prove true for $n=k+1$.

RTP

$2 \times 1! + 5 \times 2! + \dots (k^2+1)k! + ((k+1)^2+1)(k+1)!$

$= (k+1)(k+2)!$

$LHS = k \times (k+1)! + (k^2+2k+2)(k+1)!$

$= (k+1)! \left[k + k^2 + 2k + 2 \right]$

$= (k+1)! (k^2+3k+2)$

$= (k+1)! (k+2)(k+1)$

$= (k+2)! (k+1)$

$= RHS$

$\therefore$ true for $n=k+1$.

c) $f(x) = 6x - 2x^3$

$f'(x) = 6 - 6x^2$

let $f'(x) = 0$

$6 - 6x^2 = 0$

$\therefore x = \pm 1$

$f''(x) = -12x$

$f''(1) = -12$ $(1, 4)$ is a max t.e.

$f''(-1) = 12$ $(-1, -4)$ is a min t.e.

Domain: $\{x : -1 \leq x \leq 1\}$.

(ii) $y = 6x - 2x^3$

$\frac{dy}{dx} = 6 - 6x^2$

$\therefore$ at $x = 0$

$\frac{dy}{dx} = 6$

$\frac{dx}{dy} = \frac{1}{6}$

$\therefore m = \frac{1}{6}$

d)(i) $\angle ABC = (360° - 329°) + 49°$

$= 80°$

(ii) $\tan 16° = \dfrac{800}{BC}$

$\therefore BC = \dfrac{800}{\tan 16°}$

$\tan 23° = \dfrac{800}{AB}$

$AB = \dfrac{800}{\tan 23}$

$(AC)^2 = \left(\dfrac{800}{\tan 16}\right)^2 + \left(\dfrac{800}{\tan 23}\right)^2$

$- 2\left(\dfrac{800}{\tan 16}\right)\left(\dfrac{800}{\tan 23}\right)\cos 80°$

$= 9509760$

$\therefore AC = 3083.79$

$= 3084 m$

13. a) (i) $y = \frac{x^2}{4}$ $y' = \frac{x}{2}$

$m_T = \frac{2p}{2}$

$= p$

$m_N = -\frac{1}{p}$

$y - p^2 = -\frac{1}{p}(x - 2p)$

$py - p^3 = -x + 2p$

$x + py - 2p - p^3 = 0$

(ii) let $x = 0$

$py - 2p - p^3 = 0$

$py = 2p + p^3$

$y = 2 + p^2$

$Q(0, 2 + p^2)$

$M \rightarrow \frac{0 + 2p}{2}, \frac{p^2 + p^2 + 2}{2}$

$(p, p^2 + 1)$

(iii) $\therefore y = x^2 + 1$

b) $a = 2x^3 + 2x$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$$

$$\frac{1}{2}v^2 = \frac{1}{2}x^4 + x^2 + c$$

when $x = 0$ $v = 1$.

$$\frac{1}{2} = c$$

$$\frac{1}{2}v^2 = \frac{1}{2}x^4 + x^2 + \frac{1}{2}$$

$$v^2 = x^4 + 2x^2 + 1$$

$$= (x^2 + 1)^2$$

$$v = \pm(x^2 + 1)$$

but when $x = 0$ $v = 1$.

$\therefore$ $v = x^2 + 1$.

(ii) $\frac{dx}{dt} = x^2 + 1$.

$$\frac{dt}{dx} = \frac{1}{x^2 + 1}.$$

$$t = \tan^{-1}(x) + c$$

$x = 0$, $t = 0$ $\therefore c = 0$.

$\therefore$ $x = \tan(t)$.

$x = \tan\left(\frac{\pi}{4}\right)$

$= 1\,m.$

c) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= \frac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} 1 + \cos 2x \, dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right.$

$= \frac{\pi}{4}.$

d) width $= x$ length $= 2x$

$A = 2x^2$ $\frac{dA}{dt} = 18$

$\frac{dA}{dx} = 4x$

$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$18 = 4x \times \frac{dx}{dt}$

$x = 80$

$$\frac{dx}{dt} = \frac{18}{320}$$

$$= \frac{9}{160}$$

$$P = 6x$$

$$\frac{dP}{dx} = 6$$

$$\frac{dP}{dt} = \frac{dx}{dt} \times \frac{dP}{dx}$$

$$= \frac{9}{160} \times 6$$

$$= \frac{27}{80} \text{ cm/s}$$

14. a) $T = 20 - Be^{-kt}$

$$\frac{dT}{dt} = -Be^{-kt} \times -k$$

$$= kBe^{-kt}$$

$$T = 20 - Be^{-kt}$$

$$Be^{-kT} = 20 - T$$

$$\therefore \frac{dT}{dt} = k(20 - T)$$

(ii) $t = 0$, $T = -10$

$$-10 = 20 - Be^{0}$$
$$-30 = -B$$

$$\therefore B = 30$$

(iii) $3 = k(20 + 10)$

$$\therefore k = \frac{3}{30}$$

$$3 = k(20 + 10)$$

$$k = \frac{1}{10}$$

(iv) $T = 20 - 30e^{-\frac{1}{10}\times 5}$

$= 1.8\ldots$

$= 2^\circ$.

b)(i) $x = vt\cos\alpha$

$\therefore t = \dfrac{x}{v\cos\alpha}$.

$$y = v\left[\frac{x}{v\cos\alpha}\right]\sin\alpha - \frac{1}{2}g\left[\frac{x}{v\cos\alpha}\right]^2$$

$$= x\tan\alpha - \frac{gx^2}{2v^2}\sec^2\alpha$$

(ii)

$$1.8 = 10\tan\alpha - \frac{10(10)^2}{2(20)^2}\sec^2\alpha$$

$$1.8 = 10\tan\alpha - \frac{1000}{800}(1+\tan^2\alpha)$$

$$1.8 = 10\tan\alpha - 1.25 - 1.25\tan^2\alpha$$

$$1.25\tan^2\alpha - 10\tan\alpha + 3.05 = 0$$

$$\tan\alpha = \frac{10 \pm \sqrt{100 - 4(1.25)(3.05)}}{2.5}$$

$=$

$\therefore \tan\alpha = 7.682390528$

$\alpha = 82.58\ldots$

$= 83°$

or

$\tan\alpha = 0.3176094721$

$\alpha = 17.62\ldots$

$= 18°$

c)

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \dots \binom{n}{n}x^n$$

$$\int_0^3 (1+x)^n\,dx = \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^3$$

$$= \left(\frac{4^{n+1}}{n+1}\right) - \left(\frac{1^{n+1}}{n+1}\right)$$

$$= \frac{1}{n+1}\left(4^{n+1} - 1\right)$$

$$\int_0^3 \binom{n}{0}x^0 + \binom{n}{1}x^1 + \dots \binom{n}{n}x^n\,dx$$

$$= \left[\binom{n}{0}x^1 + \binom{n}{1}\frac{x^2}{2} + \dots \binom{n}{n}\frac{x^{n+1}}{n+1}\right]_0^3$$

$$= \binom{n}{0}3^1 + \binom{n}{1}\frac{3^2}{2} + \dots \binom{n}{n}\frac{3^{n+1}}{n+1}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{3^{k+1}}{k+1}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}3^{k+1}$$

$$\therefore \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}3^{k+1} = \frac{1}{n+1}\left(4^{n+1}-1\right)$$